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1997 J. Phys. A: Math. Gen. 30 2287

(http://iopscience.iop.org/0305-4470/30/7/011)

J. Phys. A: Math. Gen. **30** (1997) 2287–2296. Printed in the UK PII: S0305-4470(97)78846-2

Stochastic escape processes from a non-symmetric potential normal form II: the marginal case

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Received 18 October 1996

Abstract. The first-passage time distribution to reach the attractor of the stochastic differential equation $\dot{X}(t) = a(X^2 - X^3) + \sqrt{\epsilon}\xi(t)$ is analytically obtained by using a previously reported scheme: *the stochastic path perturbation approach*. A second-order perturbation theory, in the small noise parameter $\sqrt{\epsilon}$, is introduced to analyse the random escape, of the stochastic paths, from the marginal unstable state $X = 0$. The anomalous fluctuation of the phase-space variable $X(t)$ is analytically calculated by using the instanton-like approximation. We have carried out Monte Carlo simulations showing good agreement with our theoretical predictions.

1. Introduction

Nonlinear systems far from equilibrium exhibit a variety of instabilities when the appropriate control parameters are changed [1, 2]. By such changes of the control parameters the system can be placed in an unstable state. Therefore the system, in general, will relax to a metastable (or global) stationary state. This transient process is triggered by the noise $\mathcal{O}(\sqrt{\epsilon})$ and the statistical description of such a transient constitutes one of the main subjects of non-equilibrium statistical mechanics.

A detailed description of the relaxation process depends on the nature of the normal form involved near the critical point of the system. Typical cases are those possessing the inversion symmetry transformation $X \to -X$. This case has been studied in order to analyse the on-resonance single-mode laser with saturable absorber and in the optical Freedericksz transition [3, 4]. The theoretical approach is based on the fact that each stochastic path (up to $\mathcal{O}(\sqrt{\epsilon})$) can be approximated systematically with a suitable perturbation on the deterministic one. Therefore the lifetime of an unstable state can be studied in terms of the random escape times, which in fact are governed by those approximated stochastic paths. This fact allows us, in principle, to find—analytically—the lifetime of any unstable state. The lack of an initial Gaussian regime does not pose any restrictions for determining the statistical properties of the lifetime from an unstable state. Even for the case where the inversion symmetry does not hold, the theory of the *stochastic path perturbation approach* (SPPA) has shown to be a powerful technique to find an approximation to the first-passage time distribution (FPTD) [5].

In our previous works we have pointed out that the time-scale characterizing the escape from the instability is the lifetime of the unstable state calculated as the mean first-passage time (MFPT) [3, 5, 6]. Also the study of the transient relaxation of the system, i.e. the anomalous fluctuations in the phase-space variable can be calculated using the SPPA.

Of particular importance is the case when the potential, in the normal form, breaks the inversion symmetry $X \to -X$ and the unstable state is marginal at $X = 0$. Here we will focus on that special case. Let us rescale the order parameter in such a way that the attractor of the system is located at $X_{at} = 1$. Our physical motivation to study the stochastic differential equation (SDE)

$$\frac{\mathrm{d}}{\mathrm{d}t}X = b + a(X^2 - X^3) + \sqrt{\epsilon}\xi(t) \tag{1}$$

is inspired on the stochastic Semenov model for thermal explosive systems [6, 7], but this kind of normal form (1) also appears in the standard model for the purely absortive optical bistability laser [8]; the marginal case corresponds to $b = 0$. In the SDE (1) a is a positive constant, $\xi(t)$ is a zero mean Gaussian white noise and X represents the order parameter of the system near the critical point.

The FPTD from $X = 0$ to reach the attractor of (1) is given in terms of the lifetime of the marginal unstable state, which is characterized by the SDE

$$\frac{\mathrm{d}}{\mathrm{d}t}X = aX^2 + \sqrt{\epsilon}\xi(t). \tag{2}$$

This SDE can be worked out in a similar way as in [5], but we should remark that due to the flatness of the potential at the initial stage (i.e. the marginal unstable point $X = 0$) the stochastic trajectories which go to the left will be strongly influenced by the 'repulsive' potential wall, while in contrast those trajectories going to the right can be well approximated by biased Wiener stochastic paths. This is why a first-order perturbation theory in $\sqrt{\epsilon}$ is not enough to obtain a good description of the FPTD as we remarked in our previous paper [5].

Up to a first-order perturbation in the small parameter $\sqrt{\epsilon}$, the SPPA predicts the probability measure

$$P_o(t_e) = \frac{3^{3/2}}{a\sqrt{2\pi\epsilon}t^{5/2}} \exp\left(-\frac{3}{2}(a^2\epsilon t_e^3)^{-1}\right) \tag{3}$$

which also gives the MFPT

$$\langle t_e\rangle_{P_o(t_e)} = (a^2\epsilon)^{-1/3}\Gamma(\tfrac{1}{6})(\tfrac{3}{2})^{1/2}/\sqrt{\pi} \equiv A^{-2/3}\Gamma(\tfrac{1}{6})(\tfrac{3}{2})^{1/2}/\sqrt{\pi}. \tag{4}$$

The probability measure (3) can also be obtained by taking the limit $b \to 0$ in the distribution presented in equations (17) and (18) of [5]. However, the FPTD (3) does not give a good description of the random escape trajectories from the marginal unstable point $X = 0$.

Relaxation from a marginal unstable state has several interesting features that make it very different from relaxation from an unstable state. Typical experiments concerning relaxation close to marginality appear in optical bistable devices [8]. Strictly speaking at the marginal unstable state, fluctuations are necessary to leave the state $X = 0$ [5].

In this paper we shall present a much better approximation for tackling this problem. Thus, in principle will be able to analyse all the moments of the first-passage time. We have also made a comparison with Monte Carlo simulations showing excellent agreement with our theoretical predictions.

The paper is organized as follows. In section 2 we develop a second-order perturbation theory which naturally introduces a set of two (non-independent) random numbers, therefore the application of the SPPA allows us to calculate the FPTD from $X = 0$ to the attractor

$X_{at} = 1$ of the problem posed in (1). In section 3 we introduce the instanton-like approximation in order to study the anomalous fluctuation of the order parameter $X(t)$, and in section 4 we present the conclusions and our future research programme; detailed calculations of the probability measure can be found in the appendix.

2. The stochastic path perturbation approach

2.1. Second-order perturbation

The problem presented in this paper is the characterization of the random time when the stochastic process (1) reaches—for the first time—the attractor $X_{at} = 1$ (by looking at each stochastic realization of equation (2)). In this way we shall define a random escape time, t_e, as the random time when amplitude $X_{sppa}(t)$ diverges. This means that the time, t_e, is going to be a function of a set of random numbers which ultimately will be characterized by a specific probability measure. Then, the probability distribution of t_e, i.e. the FPTD is going to be characterized in a close way.

In order to introduce a perturbation theory in $\sqrt{\epsilon}$ it is convenient to use the parameter $A = a\sqrt{\epsilon}$, see equation (4). Following [5] we can write the stochastic paths, $X_{sppa}(t)$, as the ratio of two stochastic processes

$$X_{sppa}(t) = \frac{H(t)}{Y(t)}. \tag{5}$$

Using this nonlinear transformation in equation (2) we obtain an equivalent set of coupled equations†

$$\frac{\mathrm{d}}{\mathrm{d}t} H(t) = \sqrt{\epsilon} Y(t)\xi(t) \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} Y(t) = -aH(t) \tag{7}$$

where

$$\langle \xi(t) \rangle = 0 \qquad \text{and} \qquad \langle \xi(t)\xi(t') \rangle = \delta(t - t').$$

Here the initial conditions are $X(0) = H(0) = 0$ and $Y(0) = 1$‡. For small $\sqrt{\epsilon}$ an approximate solution of the coupled equations (6) and (7) can be considered approaching $Y(t)$ in equation (6). At the initial noise-diffusive regime in which $Y(t)$ is close to its initial value, $H(t)$ is essentially a 'pure' diffusion process, so we obtain

$$H(t) \cong \sqrt{\epsilon} W(t) \tag{8}$$

where

$$W(t) = \int_o^t \xi(t')\,\mathrm{d}t' \tag{9}$$

is the Wiener process, here $W(0) = 0$ has been used. In order to look for an iterative solution, starting with $Y(0) = 1$, we solve equation (7) with the approximate solution of $H(t)$ given by equation (8)

$$Y(t) \cong 1 - a\sqrt{\epsilon}\Omega(t) + a^2\epsilon\Theta(t) \tag{10}$$

† Note that at this point we should define a prescription for the stochastic calculus. In particular we are going to use the Stratonovich one. But the final result is independent of the specific calculus.

‡ In the absence of noise ($\epsilon = 0$), $X = 0$ is the solution for all time in agreement with the dynamics for the deterministic case of equation (2).

where the new stochastic processes $\Omega(t)$ and $\Theta(t)$ are defined by

$$\Omega(t) = \int_o^t W(t')\,\mathrm{d}t' \tag{11}$$

$$\eta(t) = \int_o^t \Omega(t')\,\mathrm{d}W(t') \tag{12}$$

$$\Theta(t) = \int_o^t \eta(t')\,\mathrm{d}t' \tag{13}$$

$\Omega(t)$ is a renormalized Gaussian process but $\Theta(t)$ is non-Gaussian. The difficulty in (10) lies on the fact that both processes are in fact correlated (see the appendix).

After introducing a scaling transformation in the Wiener integrals, a second-order approximation for the stochastic paths $X_{sppa}(t)$ can be written in the form:

$$X_{sppa}(t) \cong \frac{\sqrt{\epsilon}W(t)}{1 - a\sqrt{\epsilon}t^{3/2}\Omega + a^2\epsilon t^3\Theta} \tag{14}$$

where $\Omega \equiv \Omega(1)$ and $\Theta \equiv \Theta(1)$ are random numbers. At this level the complicated mechanism of the escape process can be noticed. At the early initial stage the numerator, a Wiener process $\mathcal{O}(\sqrt{\epsilon})$, is dominant. From this expression it is simple to observe the non-trivial fluctuations of the paths. The denominator gives the corrections, up to second order in $\sqrt{\epsilon}$, to the statistics due to the nonlinear contribution in the SDE (2), i.e. aX^2. Note that the numerator of (14) is bounded if $t \neq \infty$ with probability 1. Then the escape time, defined by $X_{sppa}(t_e) = \infty$, can be obtained as the zero of the denominator of the stochastic paths given in (14)

$$1 = At_e^{3/2}\Omega - A^2t_e^3\Theta \tag{15}$$

where, as before, $A \equiv a\sqrt{\epsilon}$.

Up to this order the SPPA gives the random escape time, t_e, as a mapping with the random numbers Ω and Θ, thus the random escape time can be found by inverting $P_A(t_e)$ as a function of Ω and Θ.

Note that $P(\Omega, \Theta)$ is a non-Gaussian probability measure (see the appendix), on the other hand in order to obtain a *simple* analytical formula for $P_A(t_e)$ we need to introduce a statistically independent approximation, therefore we assume:

$$P(\Omega, \Theta) \approx P(\Omega)P(\Theta). \tag{16}$$

An improved approximation to this *statistically independent assumption* can also be incorporated by using a quadratic non-diagonal probability measure, but we will show that this is not necessary because the independent assumption (16) is enough to predict a good agreement with the numerical simulations.

Rescaling time as $s = t'/t$ in the integrals of the Wiener process an exact expression for $P(\Omega)$ can be obtained [5]. Therefore it is possible to see that Ω is a Gaussian random variable characterized by the probability measure (see the appendix)

$$P(\Omega) = \sqrt{\frac{3}{2\pi}}\exp(-3\Omega^2/2). \tag{17}$$

The calculation of $P(\Theta)$ is a difficult task, however, the calculation of the moments of Θ are straightforward (see the appendix). Thus, using a renormalization procedure we can approximate the probability measure $P(\Theta)$ by a Gaussian one

$$P(\Theta) = \sqrt{\frac{1}{2\pi\langle\Theta^2\rangle}}\exp\left(\frac{-\Theta^2}{2\langle\Theta^2\rangle}\right) \tag{18}$$

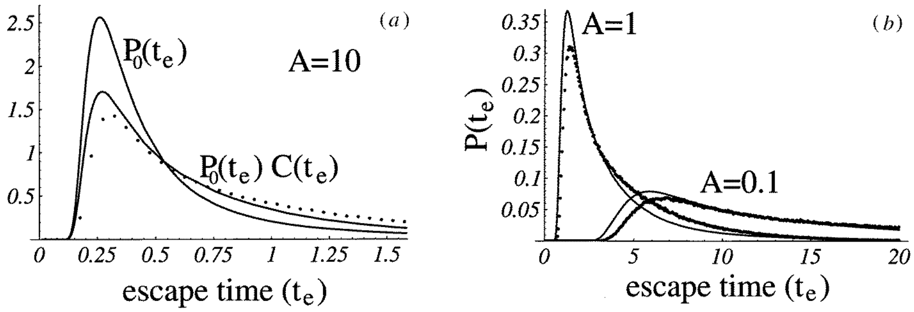

Figure 1. (*a*) Plot of FPTD $P_A(t_e) \equiv P_o(t_e)C(t_e)$, coming from the present second-order perturbation theory (19), as a function of t_e for $A = 10$. The dotted curve represents the Monte Carlo simulations of the SDE (1) with $b = 0$, having reached $X_{at} = 1$ for the first time. The corresponding $\mathcal{O}(\sqrt{\epsilon})$ probability distribution $P_o(t_e)$ is also shown. Details of the simulation are given in [5]. (*b*) Plot of FPTD $P_A(t_e)$ as a function of t_e for two values of $A(= 0.1, 1)$, the dotted curve represents the Monte Carlo simulations of the SDE (1) with $b = 0$.

where $\langle \Theta^2 \rangle = \frac{1}{180}$.

In order to work out $P_A(t_e)$ we need to look at the Jacobian of the transformation of $t_e = t_e(\Omega, \Theta)$. Thus, from (15) and after some algebraic manipulation we obtain

$$P_A(t_e) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta\left(t_e - t_e(\Omega, \Theta)\right) P(\Omega) P(\Theta)\, \mathrm{d}\Omega\, \mathrm{d}\Theta = P_o(t_e) C(t_e) \qquad (19)$$

here $P_o(t_e)$ is the $\mathcal{O}(\sqrt{\epsilon})$ contribution, already given in (3). $C(t_e)$ comes from the second-order perturbation and is given by

$$C(t_e) = \frac{\varphi\sqrt{\frac{k}{\pi}}}{2(\varphi + k)} \mathrm{e}^{\left(\frac{1}{\varphi+k}\right)} \left\{ \frac{\sqrt{\pi}\,(2\varphi + k)}{\varphi\sqrt{\varphi + k}} \left[1 + \mathrm{erf}\left(\frac{1}{\sqrt{\varphi + k}}\right)\right] + \exp\left(\frac{-1}{\varphi + k}\right)\right\} \qquad (20)$$

where

$$\varphi \equiv \tfrac{2}{3} A^2 t_e^3$$
$$k \equiv \tfrac{2}{9} \langle \Theta^2 \rangle^{-1} = \tfrac{2}{9} 180.$$

Figures 1(*a*) and (*b*) depict the $P_A(t_e)$ curves for different values of A. Also the corresponding Monte Carlo simulations are shown for the same set of parameters.

From the structure of (3) and (20) we obtain the following α-scale invariance property

$$P_{\alpha^{-3/2}A}(\alpha t_e) = \frac{P_A(t_e)}{\alpha} \qquad (21)$$

where α is any arbitrary length scale†.

The agreement between the simulations and the theory is good for different values of A, as can be seen from the short and intermediate time-domains in figures 1(*a*) and (*b*)‡. The long-time limit of $P_A(t_e)$ predicts the asymptotic behaviour $P_A(t_e \to \infty) \sim t_e^{-2.5}$ as we had reported before [5]§. Therefore the present second-order perturbation theory is an

† The parameter A under the scale transformation $X \to lX$, $t \to \alpha t$ in equation (2) goes to $\alpha^{-3/2}A$, therefore equation (21) follows.

‡ Note that from figure 1(*b*) the FPTD for $A = 1$ seems to fit better, at short times, than for the case $A = 0.1$; but the comparison should be made for the whole transient regime.

§ Note that in the physics of flames, one is usually interested in the transient behaviour of the FPTD rather than in its long-time behaviour.

important improvement to describe (analytically) the whole transient of the FPTD. This fact can be seen, in figure 1(a), from the comparison between $P_o(t_e)$ and the full expression $P_A(t_e) = P_o(t_e)C(t_e)$ given in (19) and (20).

We want to remark that a second-order perturbation theory is a fundamental necessity in order to be able to get a good probability distribution for any value of A. This is so because if we only consider $P_o(t_e)$ the error in this distribution, around the most probable value of t_e, is some time larger than the 30% for small values of A and even worse for larger values.

2.2. Moments of the FPTD

The first and second cumulants of the FPTD, $P_A(t_e)$, are

$$\langle t_e \rangle = \int_o^\infty t_e P_A(t_e)\,\mathrm{d}t_e \tag{22}$$

and

$$\langle (t_e - \langle t_e \rangle)^2 \rangle = \int_o^\infty (t_e - \langle t_e \rangle)^2 P_A(t_e)\,\mathrm{d}t_e. \tag{23}$$

From (19) those cumulants can be analytically calculated as

$$\langle t_e \rangle = F_1(A) \tag{24}$$

where

$$F_1(A) = \left(\frac{3}{2}\right)^{4/3} \frac{\sqrt{k}}{\pi} A^{-2/3} \int_0^\infty \mathrm{d}\varphi\, \frac{\varphi^{-1/6}}{(\varphi + k)} \exp\left(\frac{-k}{\varphi\,(\varphi + k)}\right) \times \left\{ \frac{\sqrt{\pi}\,(2\varphi + k)}{\varphi\sqrt{\varphi + k}} \left[1 + \mathrm{erf}\left(\frac{1}{\sqrt{\varphi + k}}\right)\right] + \exp\left(\frac{-1}{\varphi + k}\right)\right\}. \tag{25}$$

On the other hand the variance is

$$\langle (t_e - \langle t_e \rangle)^2 \rangle = (F_2(A) - F_1^2(A)) \tag{26}$$

with

$$F_2(A) = \left(\frac{3}{2}\right)^{5/3} \frac{\sqrt{k}}{\pi} A^{-4/3} \int_0^\infty \mathrm{d}\varphi\, \frac{\varphi^{1/6}}{(\varphi + k)} \exp\left(\frac{-k}{\varphi\,(\varphi + k)}\right) \times \left\{ \frac{\sqrt{\pi}\,(2\varphi + k)}{\varphi\sqrt{\varphi + k}} \left[1 + \mathrm{erf}\left(\frac{1}{\sqrt{\varphi + k}}\right)\right] + \exp\left(\frac{-1}{\varphi + k}\right)\right\}. \tag{27}$$

In figure (2) we present $\langle t_e \rangle$ and $\langle (t_e - \langle t_e \rangle)^2 \rangle$ as function of the universal parameter A, i.e. in this approximation the MFPT goes as $A^{-2/3}$, on the other hand the variance of t_e behaves as $A^{-4/3}$. This non-trivial behaviour is quite different when compared with the regular case ($b \neq 0$) treated in [5].

Note that in the small noise limit the variance of the MFPT increases, this result was expected because the FPTD gets wider and wider as soon as A decreases. Therefore if A decreases the most probable value of the FPTD lacks physical meaning because the distribution goes to a very broad probability density. We stress that using the initial condition $X(0) = 0$ our results, given in term of FPTD, go beyond the scope of the Colet *et al* [8] analysis.

Finally, we want to remark that for the initial condition $X(0) = 0$, our theoretical predictions are in good agreement with the Monte Carlo simulation as can be readily seen from figures 1(a) and (b).

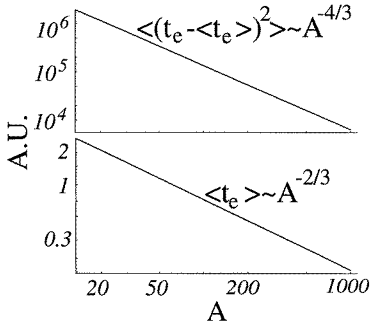

Figure 2. Log–log plot of the dimensionless mean value and variance of $P_A(t_e)$, as a function of $A \equiv a\sqrt{\epsilon}$ the universal parameter of the normal form (2). Note that the present second-order perturbation theory removes the divergency of the second moment appearing from a first-order perturbation one, see also [5].

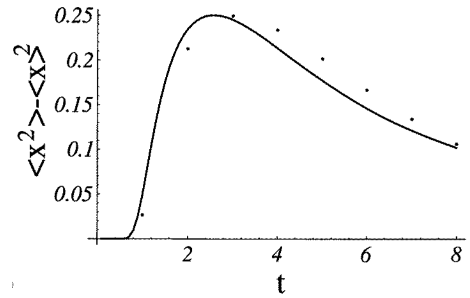

Figure 3. Plot of anomalous transient fluctuations $\sigma(t) = \langle X(t)^2\rangle - \langle X(t)\rangle^2$ from (30) as a function of t for $A = 1$, the dotted curve represents the Monte Carlo simulations of the SDE (1) with $b = 0$.

3. Transient fluctuations

In this section we basically follow our previous works [5, 3]. The transient fluctuation in the phase-space variable is just defined as the mean quadratic deviation of the $X(t)$ process [2]

$$\sigma(t) = \langle X^2\rangle - \langle X\rangle^2. \tag{28}$$

In order to calculate the anomalous fluctuation, a saturation term in the normal form equation (2) must be taken into account, i.e. we have to consider equation (1). Therefore we approximate the transient towards the global attracting solution by introducing the instanton-like approximation:

$$X(t) = X_{at}\,\Xi(t - t_e) \tag{29}$$

with X_{at} the $\mathcal{O}(1)$ macroscopic amplitude of the space variable characterizing the attractor, and $\Xi(t - t_e)$ the heaviside step function. Thus, the transient anomalous fluctuation is characterized by

$$\sigma(t) = \Lambda(t)(1 - \Lambda(t)) \tag{30}$$

where

$$\Lambda(t) = \langle \Xi(t - t_e)\rangle = \int_o^\infty \Xi(t - t_e)P_A(t_e)\,\mathrm{d}t_e = \int_o^t P_A(t_e)\,\mathrm{d}t_e \tag{31}$$

where $P_A(t_e)$ is given in (19).

From figure 3 we see that the maximum of the function $\sigma(t)$ is at MFPT, $t = \langle t_e\rangle$, in contrast with the regular case $b \neq 0$ (see [5]) where the maximum of the anomalous fluctuation was centred around the deterministic escape time $\tau = \sqrt{\frac{2}{ab}}$. Note that in the marginal case ($b = 0$) the width of $\sigma(t)$ decreases with the increase of the strength of the noise, this fact can also be understood in terms of the variance of t_e.

Figure 3 depicts the agreement of the anomalous fluctuation $\sigma(t)$, for $A = 1$, against the Monte Carlo simulations. On the other hand, as is usual, in the transient regime the initial fluctuations are amplified and give rise to the transient anomalous fluctuations of

$\mathcal{O}(1)$ as compared with the initial or final phase-space fluctuations of $\mathcal{O}(\sqrt{\epsilon})$. Our formula (30) gives a good result to describe, analytically, for the marginal case, the anomalous fluctuation of the phase-space variable near the critical point.

4. Conclusion and discussion

This paper is inspired in a method recently developed and already successfully applied to study relaxation from a subcritical pitchfork bifurcation [3] and to a non-symmetric potential normal form [5]. In those previous papers some of us have shown that—at the marginal case—the stochastic Semenov model leads to the normal form (2), therefore the FPTD is the relevant distribution to study thermal explosive systems, in well stirred chemical reactors, near the critical point [7, 1, 6].

In this paper we have analytically found the FPTD from $X = 0$ to reach the attractor $X_{at} = 1$ by analysing the lifetime to leave the marginal unstable state $X(0) = 0$. In section 2 we have introduced the SPPA and we have found a second-order perturbation theory to describe those stochastic paths, see (14).

Figure 1 shows that the FPTD is in good agreement with the Monte Carlo simulation. For large $A = a^2\sqrt{\epsilon}$ the agreement is also good, even when large values of A, mean large noise, and therefore our paths (14) could start to fail.

All the moments of the first-passage times can be written in terms of the probability measure $P_A(t_e)$, also the universal parameter of the system (at the marginal case) was shown to be $A = a^2\sqrt{\epsilon}$. In this approximation the FPTD has a long-tail characterized by the power law asymptotic from $P_A(t_e) \sim t_e^{-2.5}$ for $t_e \to \infty$. Another remarkable result from our FPTD $P_A(t_e)$ is the α-scale invariance property given in (21), which becomes a useful tool to analyse experimental results. As soon as the noise decreases ($A \ll 1$) the FPTD goes to a very broad probability measure, this is so because for the marginal case the deterministic escape time diverges.

We have studied the anomalous fluctuation of (1), showing a very good agreement with Monte Carlo simulations, see figure 3.

As a final remark we conclude that another interesting phenomenon to be studied is the non-homogeneous generalization (a Ginzburg–Landau model) of (1). Near the critical point, a non-homogeneous physicochemical reactor can be studied in terms of this sort of normal form [9]. The present work and those previously reported [5] give the mathematical tools to tackle this fundamental problem. Work along this line is in progress.

Acknowledgments

MOC wishes to thank Professor G Nicolis for useful discussions on the problem of physicochemical reactors. We also thank fruitful discussion with R Toral and M A Rodriguez about Monte Carlo simulations, and to Professor V Grünfeld for the English reading of the manuscript. We would also like to express our gratitude to ANTORCHAS by the grant A-13359/1-000050 and CONICOR proyecto N: 3232/94. CEB is grateful to the Centro Atómico Bariloche for its hospitality and to Secretaríia de Ciencia y Tecnología de la Universidad Nacional de Córdoba (Argentina) for all its support. MAF thanks the fellowship from Instituto Balseiro.

Appendix

From definitions (11)–(13) it is simple to see that the processes $\Omega(t)$, $\Theta(t)$ and $\eta(t)$ fulfill the set of SDE:

$$\dot{\Theta} = \eta(t) \tag{32}$$

$$\dot{\eta} = \Omega(t)\xi(t) \tag{33}$$

$$\dot{\Omega} = W(t) \tag{34}$$

$$\dot{W} = \xi(t). \tag{35}$$

We have worked out this set of SDE in the Stratonovich sense in order to calculate

$$\langle\Theta(t)\rangle \qquad \langle\Theta(t)^2\rangle \qquad \text{and} \qquad \langle\Theta(t)^4\rangle.$$

In this way we can approximate $P(\Theta)$ as was given in (18).

The calculation of the correlation function $\langle\Omega(t_1)\Omega(t_2)\rangle$ is straightforward and gives

$$\langle\Omega(t_1)\Omega(t_2)\rangle = \frac{3t_2t_1^2 - t_1^3}{6} \tag{36}$$

for $t_1 \leqslant t_2$. This formula corrects a printing mistake made in [5]. Thus, using the generating functional of the process $\Omega(t)$ it is possible to see that $P(\Omega)$ is given by (17) as was also remarked in the appendix of [5].

In order to calculate the first moment of Θ it is useful to note from (34) that

$$\langle W(t)\Omega(t)\rangle = \frac{t^2}{2} \tag{37}$$

therefore $\langle\Theta(t)\rangle = \int_0^t \langle\eta(t')\rangle\,\mathrm{d}t'$ is null, and so all the odd moments of Θ are null. Thus, the probability measure $P(\Theta)$ must be centred around zero.

As the next step in order to approximate the probability measure $P(\Theta)$ we need to calculate from (32) the second moment of Θ. In order to do this it is useful to note that

$$\eta(t_1) = \Omega(t_1)W(t_1) - \int_0^{t_1} W(s_1)^2\,\mathrm{d}s_1. \tag{38}$$

Therefore using (38) it is possible to see that the correlation function of the process $\eta(t)$ is given by

$$\langle\eta(t_1)\eta(t_2)\rangle = \tfrac{1}{12}(\min(t_1, t_2))^4. \tag{39}$$

Thus, from (32) and (39) we finally get that the second moment of Θ is given by

$$\langle\Theta(t)^2\rangle = \frac{t^6}{180}. \tag{40}$$

To calculate—analytically—the moment $\langle\Theta(t)^4\rangle$ is a big task, but we have done it numerically finding the result: $\langle\Theta(t)^4\rangle \simeq (3.32 \pm 0.06)10^{-4}t^{12}$ in agreement with the theoretical scaling $\sim t^{12}$.

From this result it is possible to see that the probability measure $P(\Theta)$ is non-Gaussian. This is so because if $\Theta \equiv \Theta(1)$ were Gaussian we should have obtained $\langle\Theta^4\rangle = 3\langle\Theta^2\rangle^2$. From this result we can conclude that $P(\Theta)$ is wider than a Gaussian one with variance $\langle\Theta^2\rangle = \frac{1}{180}$. In order to know an estimation of this difference we can introduce a renormalization procedure. Thus, we enforce that the width $d \equiv \langle z^2\rangle$ of a Gaussian probability $P(z) \propto \exp(-z^2/2d)$, should minimize the difference with the true (non-Gaussian) probability measure $P(\Theta)$ (characterized by only two parameters $\langle\Theta^2\rangle$ and $\langle\Theta^4\rangle$).

This renormalization procedure leads to a nonlinear minimum square problem, which can be solved by finding the real solution of the algebraic polynomial of degree 3:

$$d^3 + (\tfrac{1}{18} - \tfrac{1}{3}\langle\Theta^4\rangle)d - \tfrac{1}{18}\langle\Theta^2\rangle = 0. \tag{41}$$

From the values given above it is possible to see, from (41), that there is no remarkable difference if we just write $P(\Theta)$ by using its second moment $\langle\Theta^2\rangle = \frac{1}{180}$, i.e.

$$P(\Theta) \simeq \sqrt{\frac{180}{2\pi}} \exp\left(\frac{-\Theta^2}{\frac{2}{180}}\right). \tag{42}$$

On the other hand, in order to see how good our approximation is concerning the statistical independence between Ω and Θ, we have calculated the correlator $K_{\Omega\Theta}$ defined by

$$K_{\Omega\Theta} = \left|\frac{\langle\Theta^2\Omega^2\rangle}{\langle\Theta^2\rangle\langle\Omega^2\rangle} - 1\right|. \tag{43}$$

We have found, numerically, that $\langle\Theta^2\Omega^2\rangle \approx (7.27\pm0.06)10^{-3}$, therefore using (40), (36) and (43) we obtain $K_{\Omega\Theta} \approx 2.9$, which is an estimation of the error made by using our statistically independent assumption. If we had wished we could have made a better approximation than (16) by introducing a two-dimensional (non-diagonal) Gaussian measure, with a parameter (proportional to $K_{\Omega\Theta}$) measuring the correlation between Ω and Θ. However, the agreement of (19) with the Monte Carlo simulations leads us to adopt the simplification of the *statistically independent assumption.*

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